## ANSWER KEYS TO TEST 5

## SPRINT ROUND

1. 20.
2. 2.
3. 21.
4. $\frac{7}{8}$.
5. 11 days.
6. \$100,000.
7. 2.
8. 5 p.m.
9. 60%.
10. 360.
11. 3/5.
12. 4.
13. \$840.
14. 4.
15. 2.
16. 288.
17. 126 minutes.
18. 47.
19. 15.
20. 19.36%.
21. 55.
22. $\frac{7}{75}$.
23. \$115.
24. 2019.
25. 24.
26. 216.
27. $\frac{22}{3}$.
28. 12.
29. $\frac{1+\sqrt{5}}{2}$.
30. 18.

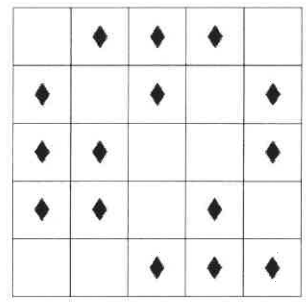
## TARGET ROUND

1. 48.
2. 20%.
3. 55.
4. 0.31.
5. 3.
6. 48.
7. B.
8. 19.

## SPRINT ROUND SOLUTIONS

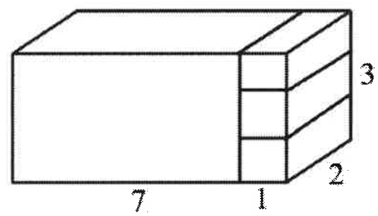
1. **Solution:** 20.
Five blue boxes weigh $6 \times 5 = 30$ ounces. Each red box weighs $30/3 = 10$ ounces and two red boxes weigh $2 \times 10 = 20$ ounces.

2. **Solution:** 2.
Notice that two rows have four crickets, so at least two crickets must move. The pair of crickets at (1, 5) and (2, 4) on the main diagonal can be moved to (5, 3) and (5, 1) as shown.



3. **Solution:** 21.
$3 \times 7 = 21$.

Or we can use the ratio of the volumes: $\frac{7 \times 3 \times 2}{2 \times 1 \times 1} = 21$.




4. **Solution:** $\frac{7}{8}$.

$7 \times \frac{1}{8} = \frac{7}{8}$.

5. **Solution:** 11 days.
Let $n$ be the number of days needed to have a total amount greater than $20.

$2^n - 1 \geq 2000$. We know that $2^{10} = 1024$ and $2^{11} - 1 = 2 \times 1024 - 1 = 2048 - 1 = 2047 > 2000$....
So the answer is 11.

6. **Solution:** $100,000.

The median is the middle one or number 60 (i.e., 59 + 1 + 59 = 119)

Since 1 + 16 + 18 + 27 = 62, the median is within that number, and the salary is $100, 000.

7. **Solution:** 2.
The perimeter of square *EFGH* is $12\sqrt{2}$ cm. The side length is $12\sqrt{2}/4 = 3\sqrt{2}$ cm. So the area of square *EFGH* is $3\sqrt{2}\times 3\sqrt{2} = 18$.

$$\frac{S_{ABCD}}{S_{EFGH}} = (\frac{AC}{EG})^2 \qquad \Rightarrow \qquad S_{ABCD} = (\frac{AC}{EG})^2 S_{EFGH} = (\frac{1}{3})^2 \times 18 = 2.$$

8. **Solution:** 5 p.m.
The time difference is 3 hours between Boston and San Francisco.
77 – 24 – 24 – 24 = 5. Five hours after 3 p.m. is 8 p.m. So the time is 8 p.m. in Boston when the train arrives in San Francisco.
When it is 8 p.m. in Boston it is 5 p.m. in San Francisco.

9. **Solution:** 60%.
We are given a chart of male and female patients and how many have each type of blood. Since we are asked to find what percent of patients with type B blood are male, we need only consider the column dealing with type B blood.

That column shows 120 males having type B blood for a total of 200 people. So the answer is 120/200 = 60%.

10. **Solution:** 360.
Method 1:

Since 6 pigs can exchange for 2 cows, each cow can exchange for 3 pigs.

Three pigs can exchange for 9 goats. So 1 pig can exchange for 3 goats and 1 cow can exchange for 9 goats. Since 32 rabbits can exchange for 4 goats, 1 goat can exchange for 8 rabbits.

Five cows can exchange for 45 goats and 45 × 8 = 360 rabbits.

Method 2:

$$32r = 4g \qquad \Rightarrow \qquad 8r = g \qquad (1)$$

$9g = 3p$ (2)

$6p = 2c \quad \Rightarrow \quad 3p = c$ (3)

Substituting (3) into (2): $9g = c \quad \Rightarrow 45g = 5c$ (4)

Substituting (4) into (1): $5c = 45g = 45 \times 8r = 360r$.

11. **Solution:** 3/5.
Rick has exactly one of each of the 50 states' U.S. quarters.
The graph shows that 12 + 4 + 1 + 5 + 2 + 2 + 4 = 30 states joined the union in the decades 1780-1849. Therefore, 30/50 = 3/5.

12. **Solution:** 4.
We write the first few terms in the sequence:
2, 3, 6, 8, 8, 4, 2, 8, 6, 8, 8, 4, 2, 8, 6, 8, 8, 4, 2, 8, 6,…
The pattern is repeated after the third term: 8, 8, 4, 2, 8, 6.
$2016 = 3 + 6 \times 335 + 3$
$2016^{th}$ term is the same as the third term in the pattern, 4.

13. **Solution:** $840.
Debra has $\frac{5}{3+4+5} \times 2016 = 840$.

14. **Solution:** 4.
The weight of $A$ and $B$ is (8 + 4 + 4 + 1) + (8 + 8 + 4 + 4 + 1) = 17 + 25 = 42.

We see that 42 + 1 = 27 + 8 +8.

So 4 weights (1, 8, 8, 27) will do.

We also see that 42 + 8 + 4 = 27 + 27

So 4 weights (8, 4, 27, 27) will also do. However, we are not able to get a smaller number less than 4. Thus the answer is 4.

15. **Solution:** 2.

$a^2+b^2+a^2b^2=4ab-1 \quad \Rightarrow \quad (a^2-2ab+b^2)+(ab)^2-2ab+1=0 \quad \Rightarrow$

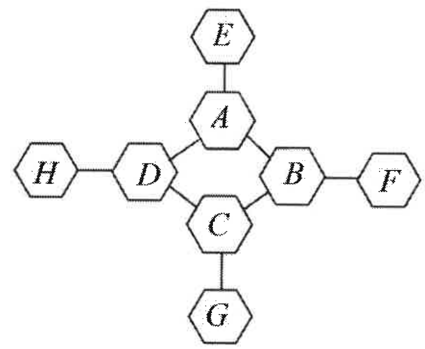
$(a-b)^2+(ab-1)^2=0$.

Since $a$ and $b$ are real numbers, we have $(a-b)^2=(ab-1)^2=0$.

So $a = b$ and $ab = 1$.
$b^2=1 \quad \Rightarrow \quad a = b = 1$ or $a = b = -1$ (ignored since we want the positive value).

The answer is 1 + 1 = 2.

16. **Solution:** 288.
Since the middle hexagons have more line segments connections, we color them first.
We have three ways to color the hexagon A (we have three colors to use to color the hexagon A), two ways to color B. If C is colored the same color as A, we have two ways to color D; If C is colored the different color from A, we have one way to color D.




So we have $3 \times 1 \times 2 + 3 \times 2 \times 2 = 6 + 12 = 18$.
We have 2 ways to color each of E, F, G, and H.
So the answer will be $18 \times 2^4 = 288$.

17. **Solution:** 126 minutes.
Method 1:
Let $x$ denote $B$'s time in minutes. Letting $l$ represent the length of the track, we have
$\frac{1}{70}(45)+\frac{1}{x}(45)=1 \quad \Rightarrow x = 126$.

Method 2:

Let $t$ be the time needed to run around the track. Then we have $\frac{45}{70-45}=\frac{t}{70}$.
Solving for $t$ gives us $t = 126$ minutes.

Method 3:
Let $d$ be the length of the path.
Since Alex can complete a circular path in 70 minutes, we have

$d = r_A t = 70r_A$ (1)

Since they meet every 45 minutes, we have

$d = (r_A + r_B)t = 45(r_A + r_B)$ (2)

Solving (1) and (2), we get:

$$r_B = \frac{5}{9}r_A \quad \Rightarrow \quad r_B = \frac{5}{9}r_A \times \frac{70}{70} = \frac{1}{126}d \Rightarrow \quad d = 126r_B$$

So it takes 126 minutes for Bob to complete the circular path.

18. **Solution:** 47.

From the top and bottom, we see $5.5 \times 2 = 11$ squares. From the left and right side, we see $4 \times 2 = 8$ squares. From the front and back, we see $14 \times 2 = 28$ squares.

The answer is then $11 + 8 + 28 = 47$.

19. **Solution:** 15.

Sam travelled $80 - 50 = 30$ miles in two hours from 3 p.m. to 5 p.m.
The average speed is $30/2 = 15$ mph.

20. **Solution:** 19.36%.

Let $x$ be the cost of the stock on Monday morning. At the end of Monday, it has lost 10% of its price or $0.1x$. So the remaining value will be $0.9x$.

At the end of the day on Thursday, the remaining value is
$1.6 \times 0.7 \times 0.8 \times 0.9 \times x = 0.8064x = (80.64\%)x$.

The overall percent loss in value from the beginning of Monday to the end of Thursday is 100% - 80.64% = 19.36%.

21. **Solution:** 55.

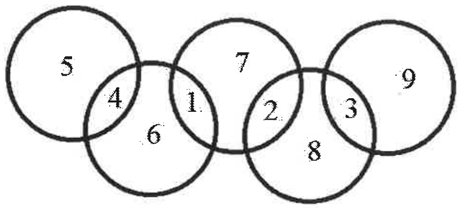
Five sums are: $S = (A + B) + (B + C + D) + (D + E + F) + (F + G + H) + (H + I) = (A + B + C + D + E + F + G + H + I) + (B + D + F + H) = 45 + (B + D + F + H)$

$B$, $D$, $F$, and $H$ count twice and other letters count once.

Since we want the smallest value of $S$, we let $B + D + F + H = 1 + 2 + 3 + 4 = 10$.

The answer is $45 + 10 = 55$.

One example is




22. **Solution:** $\frac{7}{75}$.

Case 1: the number contains three 3's.

| | |
|---|---|
| We have one such number: 333 | 1 |

Case 2: the number contains two 3's.
We have

| | |
|---|---|
| 330 | 2 |
| 336 | 3 |
| 339 | 3 |

Case 3: the number contains one 3.
Let the other two digits be $a$ and $b$.
We have

| | | |
|---|---|---|
| $a + b = 0$ | (300) | 1 |
| $a + b = 3$ | (~~330,~~ 321) | 6 |
| $a + b = 6$ | (360, 351, 342, ~~333~~) | 4 + 6 + 6 |
| $a + b = 9$ | (390,381, 372,~~363,~~ 354) | 4 + 6 + 6 + 6 |
| $a + b = 12$ | (~~393,~~384, 375, 366) | 6 + 6 + 3 |
| $a + b = 15$ | (396, 387) | 6 + 6 |
| $a + b = 18$ | (399) | 3 |

Total: 1 + 2 + 3 + 3 + 1 + 6 + 16 + 22 + 15 + 12 + 3 = 84.

The probability is $P = \frac{84}{900} = \frac{7}{75}$.

23. **Solution:** $115.
Method 1:

Suppose the \$100 bill is not fake. In the end, Betsy lost 45 – 30 = \$15 only. Now the bill is fake, so she lost 15 + 100 = \$115.

Method 2:
The money exchanged between Betsy and her friend Cathy does not make any difference. The only loss is due to the stranger who took away the pair of shoes (\$45) and \$70 change, which sum to 45 + 70 = \$115.

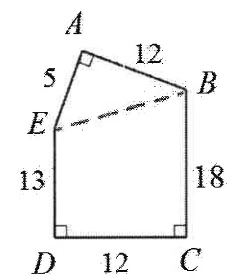
24. **Solution:** 2019.
Consider 201 and 4. 4 is not a divisor of 201. Consider 201 and 5. 5 is not a divisor of 201. Similarly, neither 2017 not 2018 is the answer. If we move on to 2019. 201 and 9 have the common factor of 3. So 2019 will satisfy our requirements.

25. **Solution:** 24.
Let the two number be $2n + 1$ and $2n + 3$.

$$(2n+1)^2+(2n+3)^2=290 \Rightarrow 8n^2+16n+10=290 \Rightarrow 8n^2+16n-280=0 \Rightarrow n^2+2n-35=0 \Rightarrow (n-5)(n+7)=0.$$

$n = 5$ and the sum of two numbers is 11 + 13 = 24.

26. **Solution**: 216.
Method 1:
Connect *BE*.
*ABE* is a right triangle and *BCDE* is a trapezoid.

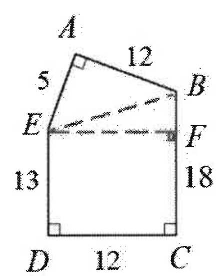



The area of the pentagon is

$$S_{\Delta ABC}+S_{BCDE}=\frac{5\times 12}{2}+\frac{(13+18)\times 12}{2}=30+186=216$$

Method 2:
Connect *BE*. Draw *EF* perpendicular to *BC* at *F*. $CF = DE = 13$.

So $BF = 18 - 13 = 5$.




Both *ABE* and *BEF* are 5-12-13 right triangles and *CDEF* is a rectangle.

The area of the pentagon is

$$2S_{AABE} + S_{CDEF} = 2\times\frac{5\times 12}{2} + 13\times 12 = 60 + 156 = 216.$$

27. **Solution:** $\frac{22}{3}$.

Let the rate for Alex be $r_a = 1/6$ and the rate for Bob be $r_b = 1/10$.

The job done in every two hours is $1\times\frac{1}{6} + 1\times\frac{1}{10} = \frac{4}{15}$.

So the fraction part of job done is $\frac{4}{15} + \frac{4}{15} + \frac{4}{15} = \frac{12}{15} = \frac{4}{5}$ in $2 \times 3 = 6$ hours.

The job left after 6 hours of working is $1 - \frac{4}{5} = \frac{1}{5}$.

When Alex works on the job for one more hour, the job left is $\frac{1}{5} - 1\times\frac{1}{6} = \frac{6-5}{30} = \frac{1}{30}$.

Bob needs $t$ hour to finish the job: $t\times\frac{1}{10} = \frac{1}{30} \quad \Rightarrow \quad t = \frac{10}{30} = \frac{1}{3}$.

So the answer is $6 + 1 + \frac{1}{3} = \frac{22}{3}$ hours.

28. **Solution:** 12.

Method 1:

We notice something special about the numbers 28, 30, 31, and 365. It is a year with 28 days in February, 30 days in April, June, September, November, and 31 days in January, March, May, July, August, October, and December.

So $x = 1$, $y = 4$, and $z = 7$. $x + y + z = 12$ (months).

Method 2:

$28x + 30y + 31z = 365$.

$28(x + y + z) + y + 2z > 29(x + y + z)$

$\Rightarrow \quad 365 > 28(x + y + z) \Rightarrow (x + y + z) < 13.03$ (1)

$31(x + y + z) - 2x - y < 31(x + y + z)$

$\Rightarrow \quad 365 < 31(x+y+z) \Rightarrow (x+y+z) > 11.77$ (2)

From (1) and 2), we get $11.77 < x+y+z < 13.03$ or $12 \le x+y+z \le 13$

Since *x, y*, and *z* are positive integers, $x+y+z = 12$ or 13.

When $x+y+z = 13$, $28x + 30y + 31z = 365 \quad \Rightarrow$

$28(x+y+z) + y + 2z = 365 \quad \Rightarrow \quad y + 2z = 365 - 28 \times 13 = 1$ (3)

Since both *y* and *z* are positive integers, (3) has no solution.

Thus $x+y+z$ must equal 12.

29. **Solution:** $\dfrac{1+\sqrt{5}}{2}$.

Let three sides be *a, b*, and *c*. *c* is the hypotenuse.

We have

$a^2 + b^2 = c^2$ (1)

$\dfrac{a}{b} = \dfrac{b}{c} \quad \Rightarrow \quad b^2 = ac$ (2)

Substituting (2) into (1): $a^2 + ac = c^2$ (3)

We divide each term of (3) by $a^2$: $1 + \dfrac{c}{a} = (\dfrac{c}{a})^2$ (4)

Let $m = \dfrac{c}{a}$, (4) becomes: $m^2 - m - 1 = 0$

$$\Rightarrow \quad m = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-1)}}{2} = \frac{1+\sqrt{5}}{2} \quad (\frac{1-\sqrt{5}}{2} \text{ ignored}).$$

30. **Solution:** 18.

The first gear has its mark face north every $\dfrac{60}{23\frac{1}{3}} = \dfrac{60}{\frac{70}{3}} = \dfrac{18}{7}$ seconds.

The second gear has its mark face north every $\dfrac{60}{40} = \dfrac{3}{2}$ seconds.

The third gear has its mark face north every $\dfrac{60}{50} = \dfrac{6}{5}$ seconds.

We know that $[\dfrac{a}{b}, \dfrac{c}{d}] = \dfrac{ac}{gcf(ad, bc)}$. We have

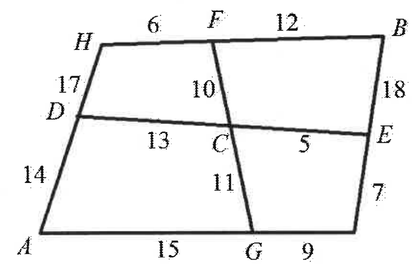
$[\frac{18}{7}, \frac{3}{2}] = \frac{18 \times 3}{gcf(18 \times 2, 7 \times 3)} = \frac{18 \times 3}{3} = 18$ seconds.

$[\frac{18}{1}, \frac{6}{5}] = \frac{18 \times 6}{gcf(18 \times 5, 1 \times 6)} = \frac{18 \times 6}{6} = 18$ seconds.

The answer will then be: 18 seconds.

## TARGET ROUND SOLUTIONS

1. **Solution:** 48.
In order to reach $B$, the player needs to go either through $F$ or $E$.
We calculate the number of hours needed from $A$ to $F$.
$ADF$: 37
$ADCF$: 37
$AGCF$: 36.
We calculate the number of hours needed from $A$ to $E$.
$AGE$: 31
$AGCE$: 31
$ADCE$: 32.
Then we compare both routes:
36 + 12 = 48
31 + 18 = 49.
The answer is 48.




2. **Solution:** 20%.
Let $a$ be the amount of salt, $m$ be the ammount of solution, and $b$ be the amount of water in the cup. We want to find $\frac{a}{m}$.

After the first addition, we have $\frac{a}{m+b}=\frac{15}{100} \Rightarrow \frac{m+b}{a}=\frac{100}{15}$

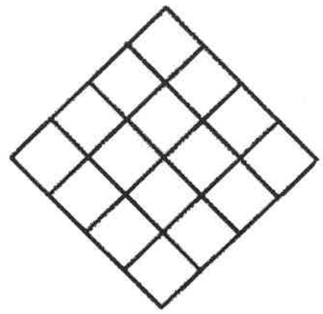
$$\Rightarrow \frac{m}{a}+\frac{b}{a}=\frac{20}{3} \Rightarrow \frac{b}{a}=\frac{20}{3}-\frac{m}{a} \qquad (1)$$

After the second addition, we have $\frac{a}{m+b+b}=\frac{12}{100} \Rightarrow \frac{a}{m+2b}=\frac{3}{25} \Rightarrow$

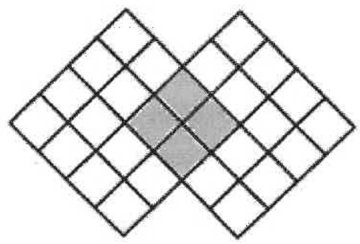
$$\frac{m}{a}+\frac{2b}{a}=\frac{25}{3} \Rightarrow \frac{2b}{a}=\frac{25}{3}-\frac{m}{a} \qquad (2)$$

$$(1)\times 2-(2):\ 0=\frac{40}{3}-\frac{25}{3}-\frac{2m}{a}+\frac{m}{a} \Rightarrow \quad 5=\frac{m}{a} \Rightarrow \quad \frac{a}{m}=\frac{1}{5}=20\%.$$

3. **Solution:** 55.
We count $4^2 + 3^2 + 2^2 + 1^2 = 30$ squares for the figure below:



We count 5 squares belonging to both figures:



By Principle of Inclusion and Exclusion, $n = n(A) + n(B) - n(A\&B)$, we get the answer $n = 30 + 30 - 5 = 60 - 5 = 55$ squares.

4. **Solution:** 0.31.

In isosceles right triangle *ABC*, $AB = 2$, $BC = 2$, and $AC = 2\sqrt{2}$.

The radius of the circle is $r = \dfrac{AB + BC - AC}{2} = \dfrac{2+2-2\sqrt{2}}{2} = \dfrac{4-2\sqrt{2}}{2} = 2-\sqrt{2}$.

The area of the circle is $\pi r^2 = \pi(2-\sqrt{2})^2$

The shaded area is $S_{\Delta ABC} - \pi r^2 = 2 - \pi(2-\sqrt{2})^2$.

The area of triangle $ACE = \dfrac{1}{2} S_{ABCD} = 3$.

The answer is $\dfrac{2-\pi(2-\sqrt{2})^2}{3} = 0.31$.

5. **Solution:** 3.

Since the geometric mean of *A*, *B* and *C* is 12, $\sqrt[3]{ABC} = 12$.

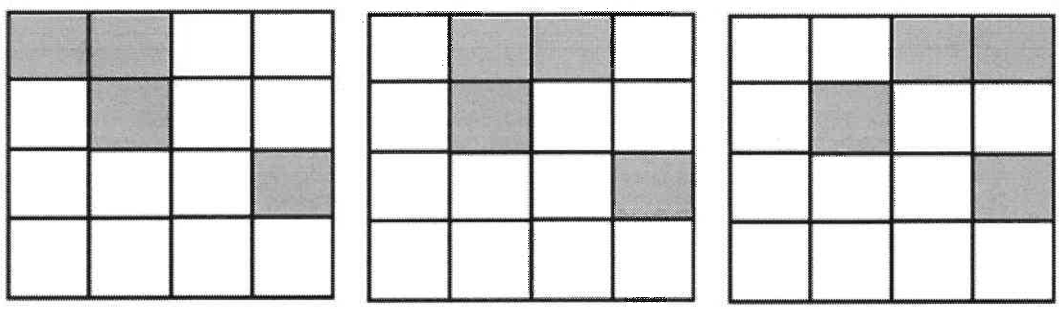
Thus $ABC = 1728$ (1)

Method 1:
We factor: $ABC = 3 \times 12 \times 48$.
So $A = 48$, $B = 12$, and $C= 3$.

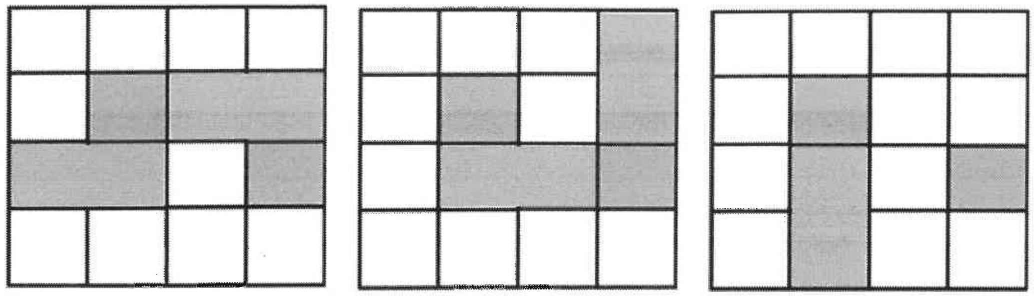
Method 2:
$A = 4B$ (2)
$B = 4C$ (3)
So $A = 16C$ (4)

Substituting (3) and (4) to (1): $(16C)(4C)C = 1728 \Rightarrow C^3 = 27 \Rightarrow C = 3$.

6. **Solution:** 48.
Method 1:
$1 \times 1$ rectangle: 14.
$1 \times 2$ rectangles:
We can count three $1 \times 2$ rectangles in each $1 \times 4$ rectangles:

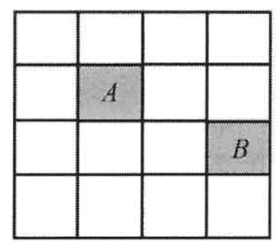


We count 5 more $1 \times 2$ rectangles as follows:



Total we have $3 \times 4 + 5 = 17$ $1 \times 2$ rectangles.
Similarly we get:
$1 \times 3$ rectangles: $2 \times 4 + 1 = 9$.
$1 \times 4$ rectangles: 4.
$2 \times 2$ rectangles: 3
$2 \times 3$ rectangle: 1
The answer is $14 + 17 + 9 + 4 + 3 + 1 = 48$.

Method 2:
The number of rectangles containing the shared area $A$:




There are just three ways to pick the lower boundary and two ways to pick the top boundary. There are 3 ways to pick the right boundary and 2 ways to pick the left boundary.

Their product is $\binom{3}{1} \times \binom{2}{1} \times \binom{3}{1} \times \binom{2}{1} = 36$

The number of rectangles containing the shared area $B$:

$$\binom{2}{1} \times \binom{3}{1} \times \binom{1}{1} \times \binom{4}{1} = 24$$

The number of rectangles containing the shared areas $A$ and $B$:

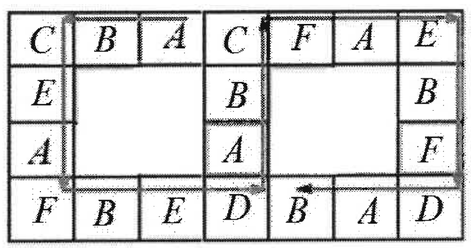
$$\binom{2}{1}\times\binom{2}{1}\times\binom{1}{1}\times\binom{2}{1}=8$$

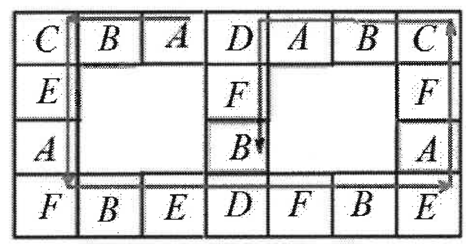
The number of rectangles containing the shared areas *A* or *B*: 36 + 24 – 8= 52.

The number of rectangles in the figure: $\binom{5}{2}\times\binom{5}{2}=100$ .

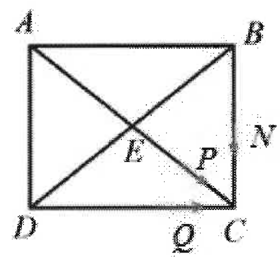
The answer is 100 – 52 = 48.

7. **Solution:** B.
We have two ways to roll the cube along the squares. The answer is *B*.




8. **Solution:** 19.
In order to reach *C*, we need to go through either *N, P,* or *Q*. So we divide the counting problem into three parts as follows:




Paths through *N:*
*ABNC,*
*AEBNC,*
*AEDABNC,*
*ADEBNC,*
*ADEABNC.*
Paths through *Q* (5 ways as well by symmetry)*:*
*ADQC,*
*AEDQC*
*AEBADQC*
*ABEDQC*

*ABEADQC*

Paths through *P*:
*AEPC,*
*AEBADEPC*
*AEDABEPC*

*ABEPC*
*ABEADEPC*
*ABEDAEPC*

*ADEPC*
*ADEABEPC*
*ADEBAEPC*
The answer is $5 + 5 + 9 = 19$.