ANSWER KEYS TO TEST 3

SPRINT ROUND

1. 43.
2. 28.
3. 147 cm.
4. 986.
5. 28.
6. $\frac{3}{11}$
7. 4889.
8. (3).
9. 12.
10. 3.
11. North.
12. 216.
13. $\frac{26}{15}$.
14. 12.
15. $55.
16. $21\frac{1}{2}$.
17. 360.
18. 4.
19. 5.
20. $n = 1/m$.
21. 12.
23. 78.
24. $\sqrt{3}$.
25. 7.
26. 330.
27. 13.
28. 495.
29. 2.
30. 8/9.

TARGET ROUND

1. 9.
2. $\frac{5}{216}$.
3. 144π.
4. $9\sqrt{3} - 4\pi$.
5. $-\frac{4}{9}$.
6. 14.
7. 20.
8. $\frac{11}{12}$.

SOLUTIONS TO TEST 3

SPRINT ROUND

1. **Solution:** 43.
$625 - 41 \times 10 = 215$
$215/5 = 43$.

2. **Solution:** 28.
A number is divisible by 4 if the last two-digit of the number is divisible 4.
So the number that is not divisible by 4 is 20174. The answer is $7 \times 4 = 28$.

3. **Solution:** 147 cm.
The average of the heights of Alex, Bob, Cathy, Debra, and Emma is $(147 + 149 + 150 + 151 + 153)/5 = 5 \times 150/5 = 150$.
So Frank's height is $150 - 3 = 147$ cm.

4. **Solution:** 986.
$1000 \div 17 = 58$ R 14
Since the remainder is 14,
$1000 - 14 = 986$.

5. **Solution:** 28.
Each column has three small squares. Each square has three ways to color. So each column there are $3 \times 3 \times 3 = 27$ ways to color. By the pigeonhole principle, we need at least $27 + 1 = 28$ columns to guarantee that at least two columns are colored exactly the same way.

6. **Solution:** $\frac{3}{11}$

Let n be the number of terms from 1 to 110 that is a multiple of 5 or 11.

$$n = \left\lfloor \frac{110}{5} \right\rfloor + \left\lfloor \frac{110}{11} \right\rfloor - \left\lfloor \frac{110}{5 \times 11} \right\rfloor = 22 + 10 - 2 = 30.$$

The probability is $P = \frac{30}{110} = \frac{3}{11}$.

7. **Solution:** 4889.
Let x be the number of the copies sold.

$1.1\ (20{,}000 + 15x) = 30x - 0.3 \times 30 \times x \quad \Rightarrow \quad x = 4888.89 \approx 4889.$

8. **Solution:** (3).

In the answers (1) and (2), D should be after E. In (4), A should be after B. In (5), we do not see D after E.

The answer is (3).

9. **Solution:** 12.

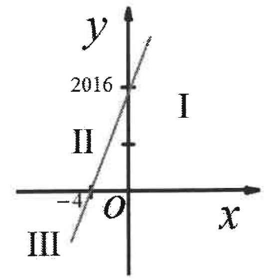
Method 1:
We list:
0.75 > 0.31, 0.75 > 0.13, 0.73 > 0.51, 0.73 > 0.15, 0.71 > 0.53, 0.71 > 0.35
0.57 > 0.31, 0.57 > 0.13, 0.53 > 0.17, 0.51 > 0.37
0.37 > 0.15, 0.35 > 0.17.
The answer is 6 + 4 + 2 = 12.

Method 2:
If the tenths digit of the first decimal is 7, no matter how we place the rest of the 3 digits, the inequality will be true. So we have 3! = 6 ways for this case.
If the tenths digit of the first decimal is 5, we have two ways (3, or 1, but not 7) to select the tenth digit of the second decimal in order for the inequality to be true. Then we have two digits left and we have two ways to order them. So we have 2 × 2 = 4 ways for this case.

If the tenths digit of the first decimal is 3, we have one way (1 only) to place the tenth digit of the second decimal in order for the inequality to be true. Then we have two digits left and we have two ways to order them. So we have 1 × 2 = 2 ways for this case.

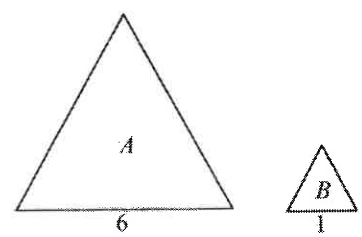
The answer is 6 + 4 + 2 = 12.

10. **Solution:** 3.
The line $y = 2016 + 504x$ is plotted using two points (0, 2016), (-4, 0) as follows:
As can be seen the line goes through 3 quadrants.



11. **Solution:** north.
$75/12 = \frac{75}{12} = \frac{25}{4} = 6\frac{1}{4}$.
The hour hand goes through $6\frac{1}{4}$ revolutions. If the hour hand started facing west, it'll end up facing north.

12. **Solution:** 216.
The ratio of the areas A to B as shown in the figure is $\frac{S_A}{S_B} = \left(\frac{6}{1}\right)^2$.
So we need 36 equilateral triangles each with side length 1 to make the larger equilateral triangles each with side length 6. To make a regular hexagon of side length 6, we need six equilateral triangles each with side length 6. So the answer is $36 \times 6 = 216$.



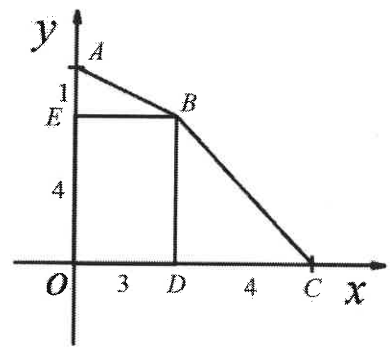
13. **Solution:** $\frac{26}{15}$.
If r, s, t are roots, by Vieta's Theorem,
$rs + st + tr = 13$
$rst = -(-15) = 15$.
The surface area is $2(rs + st + tr)$ and the volume is rst.
The ratio is $\frac{2(st + rt + rs)}{rst} = \frac{2 \times 13}{15} = \frac{26}{15}$.

14. **Solution:** 12.
By triangle inequality theorem, $11 - 7 < c < 11 + 7 \quad \Rightarrow \quad 4 < c < 18$.
Since c is an integer, the largest and smallest possible value of c are 17 and 5.

The answer is 17 – 5 = 12.

15. **Solution:** $55.
32.5 – 1 = 31.5.
31.5 ÷ (1/ 3) = 94.5
94.5 × 0.4 = 37.8
Thus the charge for the first person is: 4 + 37.8 = 41.8
The other people are $3.2 each regardless of the length of the trip.41.8 + (3.3 × 4) = 41.8 + 13.2 = $55.

16. **Solution:** $21\frac{1}{2}$.



Method 1:
The quadrilateral looks as follows:
The quadrilateral can be broken into a rectangle with width 3 units and length 4 units, 2 right triangles (ΔABE and ΔBCD).
The area is 3 × 4 + $\frac{1\times 3}{2}+\frac{4\times 4}{2}=12+\frac{3}{2}+8=21\frac{1}{2}$.
Method 2:

By the shoelace formula, the area is $A=\frac{1}{2}\begin{vmatrix}0 & 0\\ 0 & 5\\ 3 & 4\\ 7 & 0\\ 0 & 0\end{vmatrix}$

$=\frac{1}{2}\times|0+0+0+0-(0+28+15+0)|=\frac{1}{2}\times|-43|=\frac{43}{2}=21\frac{1}{2}$.

17. **Solution:** 360.
Alex's speed is 900/100 = 9 m/s. Bob's speed is 900/150 = 6 m/s.
Let t be the time taken for them to meet in every lap: $t(9+6)=900 \Rightarrow t=60$ seconds.
When they meet for the ninth time, Bob runs 9 × 60 = 540 seconds and 540 × 6 = 3240 meters.

$3240 = 900 \times 3 + 540$.
So Bob runs 3 laps and 540 meters already. He needs to run $900 - 540 = 360$ meters to reach the starting point.

18. **Solution:** 4.
We have the following prime numbers: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31.
$36 = 31 + 5 = 29 + 7 = 23 + 13 = 19 + 17$.
The answer is 4 ways.

19. **Solution:** 5.
The problem is the same as the following one:
How many ways are there to write 100 as the sum of four square numbers?

To determine what these squares could be, list all squares under 100:
1, 4, 9, 16, 25, 36, 49, 64, 81.

If 81 is chosen then three other numbers must sum to 19. We only see that $9 + 9 + 1 = 19$.

If 64 is chosen, three other numbers must sum to 36. We only see that $16 + 16 + 4 = 36$.

If 49 is chosen, three other numbers must sum to 51. We see that $49 + 1 + 1 = 25 + 25 + 1 = 51$.

If 25 is chosen, three other numbers must sum to 75. We only see that $25 + 25 + 25 = 75$.
Total we have 5 ways.

20. **Solution:** $n = 1/m$.
Let $(2\sqrt{2} - \sqrt{7})^{2016} = n$ (1)
We are given that $(2\sqrt{2} + \sqrt{7})^{2016} = m$ (2)
(1) × (2): $(2\sqrt{2} - \sqrt{7})^{2016}(2\sqrt{2} + \sqrt{7})^{2016} = m \times n \quad \Rightarrow$

$\left(\sqrt{8}-\sqrt{7}\right)^{2016}\left(\sqrt{8}+\sqrt{7}\right)^{2016}=m\times n \Rightarrow \left((\sqrt{8}-\sqrt{7})(\sqrt{8}+\sqrt{7})\right)^{2016}=m\times n \Rightarrow$
$(8-7)^{2016}=m\times n \Rightarrow m\times n=1 \Rightarrow n=1/m.$

21. **Solution:** 12.
Method 1:
We notice something special about the numbers 29, 30, 31, and 366. It is a leap year with 29 days in February, 30 days in April, June, September, November, and 31 days in January, March, May, July, August, October, and December.
So $x = 1$, $y = 4$, and $z = 7$. $x + y + z = 12$ (months).

Method 2:
$29x + 30y + 31z = 366$
$29(x + y + z) + y + 2z > 29(x + y + z) \Rightarrow$ $366 > 29(x + y + z) \Rightarrow (x + y + z) < 12.62$ (1)
$31(x + y + z) - 2x - y < 31(x + y + z) \Rightarrow$ $366 < 31(x + y + z) \Rightarrow (x + y + z) > 11.8$ (2)
From (1) and 2), we get $11.8 < x + y + z < 12.6$.
Since x, y, and z are positive integers, $x + y + z = 12$.

22. **Solution:** 533.
Let a, b, c, d, and e be the unit price for A, B, C, D, and E, respectively.
$a+3b+4c+5d+6e=2016$ (1)
$a+5b+7c+9d+11e=3499$ (2)
(1) × 2: $2a+6b+8c+10d+12e=4032$ (3)
(3) – (2): $a+b+c+d+e=533$.
Note: $a = 106$, $b = 108$, $c = 109$, $d = 110$, and $e = 100$ will work,

Or (2) – (1): $2b+3c+4d+5e=1483$ (4)
(1) – (4): $a+b+c+d+e=533$.

23. **Solution:** 78.
The first 3 squares are single digits (1, 4, 9). The next 6 squares are double digits (16, 25, 36, 49, 64, 81). The rest of 21 squares must be triple digits (100, 121, 144, 169, 196, 225, …, 900).
$3 \times 1 + 6 \times 2 + 21 \times 3 = 3 + 12 + 63 = 78$.

24. **Solution:** $\sqrt{3}$.
Let V_1 be the volume of the new tetrahedron and V_2 be the volume of the original tetrahedron. a is the length of each side. h is the height.

$$V_1 = \frac{1}{3} \times \frac{\sqrt{3}}{4} a_1^2 h_1 \quad (1)$$

$$V_2 = \frac{1}{3} \times \frac{\sqrt{3}}{4} a_2^2 h_2 \quad (2)$$

$$(1) \div (2): \frac{V_1}{V_2} = \frac{\frac{1}{3} \times \frac{\sqrt{3}}{4} \times a_1^2 h_1}{\frac{1}{3} \times \frac{\sqrt{3}}{4} \times a_2^2 h_2} = (\frac{a_1}{a_2})^2 = 3 \quad \Rightarrow \quad \frac{a_1}{a_2} = \sqrt{3}.$$

25. **Solution:** 7.
When Sally draws out a red marble, the number of red marbles will be increased by 4.
When Sally draws out a white marble, the number of white marbles will be increased by 7.

Since we want the fewest number of rounds of draws/replacements/additions, we consider more 7's possible.
We see clearly that:

R	15	4	4	4	
W	20	7	7	7	7

$15 + 20 + 4 \times 3 + 7 \times 4 = 75$. The answer is $3 + 4 = 7$.

26. **Solution:** 330.
Method 1:
The two-digit numbers are: 12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, 43. The sum is 330.

Method 2:
We have 4 ways to select the units digit and 3 ways to select the tens digit. So we have $4 \times 3 = 12$ such 2-digit numbers. We have $12 \times 2 = 24$ digits together. $24/4 =$

6. Each digit is used 6 times in these 12 numbers. Each digit is used 6/2 = 3 times in units digit position as well as the tens digit position. So we have $3\times(1+2+3+4)\times 11=330$.

27. **Solution:** 13.

$2(x-y)^3+4y^2=254 \quad \Rightarrow \quad (x-y)^3+2y^2=127 \quad \Rightarrow \quad (x-y)^3=127-2y^2$

Since 127 is odd and $2y^2$ is even, $(x-y)^3$ must be odd. We know that x and y are positive integers, so if we let $y = 1$, we get $(x-y)^3 = 127 - 2 = 125 = 5^3$.

So $x = 6$. We can also let $y = 8$, we get $(x-y)^3 = 127 - 128 = -1 = (-1)^3$. So $x = 7$.

The sum is 6 + 7 = 13.

28. **Solution:** 495.

$\frac{120}{100}(100a+10b+c)=100c+10b+a \quad \Rightarrow \quad 6(100a+10b+c)=5(100c+10b+a)$

$\Rightarrow 600a+60b+6c=500c+50b+5a \quad \Rightarrow \quad 595a+10b=494c \quad \Rightarrow$

$5(119a+2b)=494c$

Since 5 and 494 is relatively prime, c must be a multiple of 5. We know that c is a digit. So c can only be 5.

Then we have $119a+2b=494$

Since both $2b$ and 494 are even, $119a$ must be even. So the digit a must be even and it can be 2, 4, 6, or 8. Only $a = 4$ works. When $a = 4$, $b = 9$.

The original number is 495. We see that $1.2 \times 495 = 594$.

29. **Solution:** 2.

Draw $PN \perp EF$, $PM \perp CD$ (Figure 1).

We slide the hexagons $PQRSTV$ such that PN is overlapped with PV (Figure 2).

The shaded area is $\frac{4}{12}\times 6=2$.

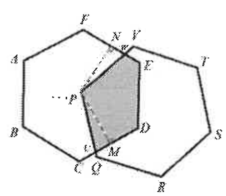


Figure 1

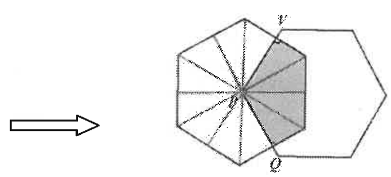


Figure 2

30. **Solution:** 8/9.

Method 1:
Let us say the names of the six teams are *A, B, C, D, E,* and *F*.

The number of ways to select 4 teams out of 6 teams is $\binom{6}{4}=\binom{6}{2}=15$.

The number of ways Team *A* being selected is: $\binom{1}{1}\binom{5}{3}=10$.

So Team *A* has a 10/15 = 2/3 probability of competing on any given day.
We have three cases where team *A* competes:

(C: compete; N: Not compete).

Day 1	Day 2	Day 3	Day 4	
C	N	N	C	$P_1=\frac{2}{3}\times\frac{1}{3}\times\frac{1}{3}\times\frac{2}{3}\times\frac{4!}{2!\times 2!}=\frac{24}{81}$

Day 1	Day 2	Day 3	Day 4	
C	N	C	C	$P_2=\frac{2}{3}\times\frac{1}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{4!}{3!}=\frac{32}{81}$

Day 1	Day 2	Day 3	Day 4	
C	C	C	C	$P_3=\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}=\frac{16}{81}$

The answer is $P_1+P_2+P_3=\frac{24}{81}+\frac{32}{81}+\frac{16}{81}=\frac{8}{9}$.

Method 2 (indirect way):
Let us say the names of the seven teams are *A, B, C, D, E,* and *F*.

The number of ways to select 4 teams out of 6 teams is $\binom{6}{4}=\binom{6}{2}=15$.

The number of ways team *A* being selected is: $\binom{1}{1}\binom{5}{3}=10$.

So team *A* has a 10/15 = 2/3 probability of competing on any given day.

We have two cases where team A is not selected more than two day of the next four days
(C: compete; N: Not compete).

Day 1	Day 2	Day 3	Day 4	
N	N	N	N	$P_1 = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{81}$
Day 1	Day 2	Day 3	Day 4	
C	N	N	N	$P_2 = \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{4!}{3!} = \frac{8}{81}$

The answer is $1-(P_1+P_2)=1-(\frac{1}{81}+\frac{8}{81})=\frac{72}{81}=\frac{8}{9}$.

TARGET ROUND SOLUTIONS

1. **Solution:** 9.

Let g be the number of green balls and b be the number of blue balls. y be the number of balls Alex takes out.

$b+g=25$

$y=\frac{b}{2}+\frac{g}{3}=\frac{b}{2}+\frac{25-b}{3}=\frac{b}{2}+\frac{25}{3}-\frac{b}{3}=\frac{b+50}{6}$.

Since we want the least value of y, we let $b = 4$ to get $y = 9$.

2. **Solution:** $\frac{5}{216}$.

Method 1:

We have two cases:

Emily rolls *ABCCC, ABAAA,* or *ABBBB,* where each letter represents a different digit.

In the first case, the probability is $P_1=1\times\frac{5}{6}\times\frac{4}{6}\times\frac{1}{6}\times\frac{1}{6}=\frac{20}{6^4}$

In the second case, the probability is $P_2=1\times\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}=\frac{5}{6^4}$.

In the third case, the probability is $P_3=1\times\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}=\frac{5}{6^4}$.

The probability is $P=P_1+P_2=\frac{20}{6^4}+\frac{5}{6^4}+\frac{5}{6^4}=\frac{5}{216}$.

Method 2:

$P=1\times\frac{5}{6}\times1\times\frac{1}{6}\times\frac{1}{6}=\frac{5}{216}$.

3. **Solution:** 144π.

$\angle APD=\frac{\overset{\frown}{AD}+\overset{\frown}{BC}}{2}=\frac{3\pi+5\pi}{2}=4\pi$.

$\angle APC=120°=2\angle APD \quad\Rightarrow\quad \angle APC=\frac{\overset{\frown}{AC}+\overset{\frown}{DB}}{2}=8\pi$

$\Rightarrow \quad \overset{\frown}{AC}+\overset{\frown}{DB}=16\pi$.

So the circumference of the circle is $16\pi + 8\pi = 24\pi$ and the radius is $2\pi r = 24\pi \Rightarrow r = 12$.
The area of the circle is $\pi \times 12^2 = 144\pi$.

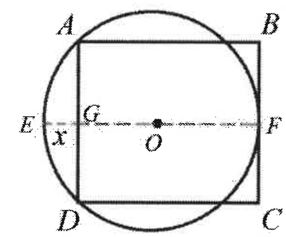
4. **Solution:** $9\sqrt{3} - 4\pi$.

Draw *EF*, the diameter from the tangent point. Let $EG = x$. We see that $GF = AB = 3\sqrt{3}$, and $AG = 6/2 = 3$.

$x \times 3\sqrt{3} = 3 \times 3 \Rightarrow \quad x = \sqrt{3}$.

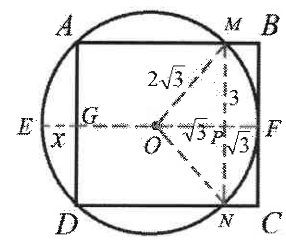
So $EF = EG + GF = \sqrt{3} + 3\sqrt{3} = 4\sqrt{3}$.

So the radius is $\frac{4\sqrt{3}}{2} = 2\sqrt{3}$, and the area of the circle is 12π.



Connect *OM, ON,* and *MN*. MP = 3, $OM = 2\sqrt{3}$, so $OP = \sqrt{3}$, and $PF = \sqrt{3}$.
Since $OM = 2OP$, right triangle *MOP* is a 30°-60°-90° right triangle. So $\angle MON = 120°$.
The area of sector $MON = \frac{1}{3} \times 12\pi = 4\pi$.

The area of sector $MPN = 4\pi - 3\sqrt{3}$.
The shaded area is then the area of rectangle *BCNM* – the area of sector $MPN = 6 \times \sqrt{3} - (4\pi - 3\sqrt{3}) = 9\sqrt{3} - 4\pi$.



5. **Solution:** $-\frac{4}{9}$.

Method 1:
We observe $a_5 = 3$, $a_2 = 1$, and $a_8 = 5$.

By the formula, $a_5 = a_2 + (5-2)d \quad \Rightarrow \quad 3 = 1 + (5-2)d \quad \Rightarrow \quad d = \frac{2}{3}$

We also know that the following terms are also work: $a_5 = 3$, $a_2 = 5$, and $a_8 = 1$.

By the formula, $a_5 = a_2 + (5-2)d \quad \Rightarrow \quad 3 = 5 + (5-2)d \Rightarrow \quad d = -\frac{2}{3}$

So the answer is $\frac{2}{3} \times (-\frac{2}{3}) = -\frac{4}{9}$.

Method 2:

We know that $a_2 = a_5 - 3d$, and $a_8 = a_5 + 3d$.

So $a_2 + a_5 + a_8 = 9 \quad \Rightarrow \quad a_5 - 3d + a_5 + a_5 + 3d = 9 \quad \Rightarrow \quad a_5 = 3$.

$a_2 \times a_5 \times a_8 = 15 \quad \Rightarrow \quad (a_5 - 3d) \times a_5 \times (a_5 + 3d) = 15 \quad \Rightarrow a_5 \times [a_5^{\,2} - (3d)^2] = 15$

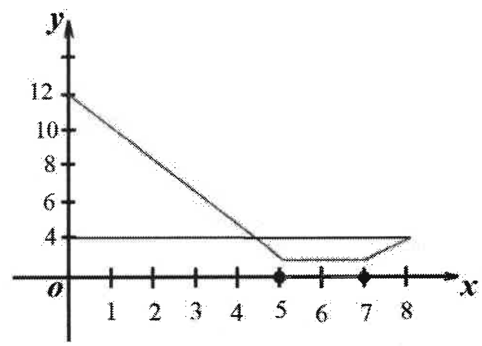
$\Rightarrow\ 9 - 9d^2 = 5 \quad \Rightarrow \quad 9d^2 = 4 \quad \Rightarrow \quad d^2 = \frac{4}{9} \quad \Rightarrow \quad d = \pm\frac{2}{3}$

So the answer is $\frac{2}{3} \times (-\frac{2}{3}) = -\frac{4}{9}$.

6. **Solution:** 14.

Method 1:

We get rid of the absolute value signs by examining three regions:



(1) $0 \le x \le 5$:
$|x-5| + |x-7| = 5 - x + 7 - x = 12 - 2x$.

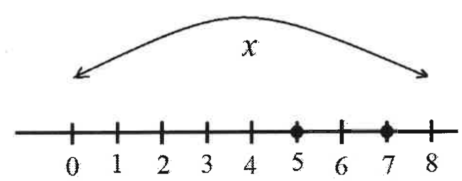
The greatest value is 12 when $x = 0$.

(2) $5 < x \le 7$:
$|x-5| + |x-7| = x - 5 + 7 - x = 2$.

(3) $7 < x \le 8$: $|x-5| + |7-x|$.

The greatest value is 4 when $x = 8$

So the answer is 12 + 2 = 14.

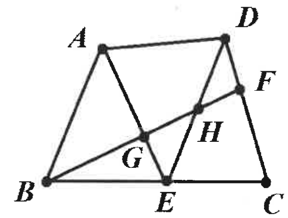


Method 2:

$|x-5| + |x-7|$ is the sum of the distances from x to 5 and to 8. The greatest distance is 5 + 7 = 12 obtained when $x = 0$. The smallest distance is 1 + 1 = 2 obtained when $5 \le x \le 7$. So the answer is 14.

Method 3: We know that $|a| + |b| \ge |a + b|$. $|x-5| + |x-7| = |x-5| + |7-x| \ge |x - 5 + 7 - x| = 2$. The smallest value is 2. We also know that the greatest value of $|x-5| + |x-7|$ is 12 when $x = 0$. So the answer is 12 + 2 = 14.

7. **Solution:** 20.



Method 1:
AB, AG, AE, AD.
BG, BH, BF, BE, BC.
CD, CF, CE.
DE, DH, DF.
EG, EH.
We get 4 + 5 + 3 + 3 + 2 + 2 + 1 = 20.
Method 2:

We get at most $\binom{8}{2} = 28$ line segment from eight points.

But we are not able to count *AF, AH, AC, BD, EF, CH, CG*, and *DG*. 28 – 8 = 20.

8. **Solution:** $\frac{11}{12}$.

There are $\binom{10}{3} = 120$ possible ways to select three numbers.

We divide ten numbers into two groups:
(1, 3, 5, 7, 9) and (2, 4, 6, 8, 10)
Only when at least one value is even can we get an even product.

We have three cases:
Case 1: Even × Even × Even.

We have the following ways to select three even numbers: $\binom{5}{3} = 10$

Case 2: Even × Even × Odd.

We have the following ways to select one even and two odd numbers: $\binom{5}{2}\binom{5}{1} = 50$

Case 3: Even × Odd × Odd.

We have the following ways to select one even and two odd numbers: $\binom{5}{1}\binom{5}{2} = 50$

The answer is $P = \frac{10+50+50}{120} = \frac{11}{12}$.